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ORIGINAL PAPER

Dynamics of pseudo-radioactive chemical products via sampling theory

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Abstract The decomposition of radioactive chemical products usually follows an exponential dynamics. The present paper deals with the problem of analyzing the unknown dynamics of pseudo-radioactive materials for which we have a temporal sample of the amount of the decomposing product. The strategy for studying if the adjustment of the dynamics is or not of exponential type is to use a recent generalization of the Shannon's sampling theorem for non-band limited signals. The aim of the paper is to present an alternative and short proof of this result with a completely different approach to the original one by using transform theory.

Keywords Pseudo-radioactive · Band-limited signal · Shannon's sampling theorem · Approximation theory

1 Introduction and statement of the main result

It is well known that the dynamics of the decomposition of the most used radioactive chemical products like Carbon 14 or Urania 238 evolves in an exponential way.

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Imagine that we have a pseudo-radioactive material with an unknown decomposition dynamics but for which we have a temporal sample of the amount of the decomposing product. In this setting for analyzing if the dynamics has or not an exponential adjustment is used a recent generalization of the Shannon Sampling Theorem for non-band limited signals [1].

A central result of the signal theory in engineering is the well-known Shannon–Whittaker–Kotel'nikov's theorem (see for instance [11] or [13]) working for band-limited maps of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener signals), and based on the normalized cardinal sinus map $\text{sinc}(t)$ defined by

$$\text{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0. \end{cases}$$

Another philosopher's stone of the signal processing theory is the Middleton's sampling theorem for band step functions (see [10]). This result was one of the first modifications of the classic Sampling theorem (see [12]) which only works for band-limited maps. After this starting point many different extensions and generalizations of this theorem appeared in the literature trying to obtain approximations of non band-limited signals (see for instance [4] or [6]). Good surveys on these extensions are [5] or [13].

[1] following the spirit of the previous results in the sense of trying to obtain approximations of non band-limited signals by using band-limited ones by increasing the band size. But this approach is completely different to the previous ones in the sense that we keep constant the sampling frequency generalizing in the limit the results of Marvasti et al. [9] and Agud et al. [2].

In this setting, [1] states and proof for Gaussian signals the following asymptotic Shannon sampling theorem type where the convergence is considered in the Cauchy's principal value for the series and point wise for the limit.

Property 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a map and $\tau \in \mathbb{R}^+$. We say that f holds the property $\mathcal{P}$ for τ if*

$$f(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}} \left(\frac{k}{\tau}\right) sinc(\tau t - k) \right)^n. \tag{1}$$

Theorem 1 *The Gaussian maps, i.e. maps of the form $e^{-\lambda t^2}$, $\lambda \in \mathbb{R}^+$ hold property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$.*

The idea of the proof presented in [1] is the following: since the Gaussian map is analytical, for proving formula (1) is enough to show the equality between the coefficients of the power series representation of the Gaussian map and the coefficients of the series stated in the second member of (1) after proving the analitycity of the second member of (1). This proof is very long, quite technical and use strongly complex analysis tools.

The aim of the present paper is to present an alternative proof to this result very short and using a completely different approach. The ideas of our proof are to use some simple facts of the transform theory and to use properly a bound given by Boas [3].

The paper is structured as follows in Sect. 2 we present the ideas and results that have inspired property $\mathcal{P}$ and in Sect. 3 we give the new proof of Theorem 1.

2 On the property $\mathcal{P}$

Property $\mathcal{P}$ is an approximation in the limit, through potentials of band-limited maps of the original signal, based on [2] and [9].

In [2] is proven that given a sequence $\{s_k\}_{k\in\mathbb{Z}} \in l^{2/n}(\mathbb{Z})$, $B > 0$, $\tau \geq 2B$ and n odd, there exist exactly n band-limited signals $\{x_r\}$ with bandwidth equal to B such that $x_r^n\left(\frac{k}{\tau}\right) = s_k$. Moreover, is shown that $x_r = e_r x_0$, where $\{e_r\}_{r=0}^{n-1}$ are the roots of unity of order n and $x_0(t) = \sum_{k\in\mathbb{Z}} s_k^{1/n}\mathrm{sinc}(2Bt - k)$.

From this is directly deduced that if we consider an odd number n and a band-limited signal f with bandwidth $\widetilde{B}$ such that the sequence of coefficients $\left\{f\left(\frac{k}{\tau}\right); k \in \mathbb{Z}\right\}$ with $\tau \geq \frac{2\widetilde{B}}{n}$ holds the properties stated in [2], then the signal admits a recomposition of Shannon type in the form

$$f(t) = \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}\left(\frac{k}{\tau}\right)\mathrm{sinc}(\tau t - k)\right)^n, \tag{2}$$

where clearly the sampling frequency can be choosen bigger than the Nyquist one.

The aim is to provide a method for approximating non band-limited signal by band-limited ones and keeping the frequency of the sampling constant. And the idea is to take limits in (2) obtaining an equality of the form

$$f(t) = \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}\left(\frac{k}{\tau}\right)\mathrm{sinc}(\tau t - k)\right)^n,$$

expressed as a property $\mathcal{P}$.

3 Reproving Theorem 1

The aim of this section is to proof Theorem 1 with a completely different approach to the original one.

Proof of Theorem 1. Boas [3] states the following estimation

$$\left|f(t) - \sum_k f(k)sinc(t-k)\right| \leq 2\int\limits_{|\xi|>1/2}\left|\widehat{f}(\xi)\right| \mathrm{d}\xi. \tag{3}$$

We assume, without loss of generality, that $\lambda = \tau = 1$. Let $f(t) = e^{-\pi t^2}$, for which $f_n(t) = f^{1/n}(t) = e^{-\pi t^2/n}$ with $\widehat{f_n}(\xi) = \sqrt{n}e^{-\pi\xi^2 n}$. By (3)

$$\left| f^{1/n}(t) - \sum_k f^{1/n}(k) sinc(t-k) \right| \leq 2\sqrt{n} \int\limits_{|\xi|>1/2} e^{-\pi\xi^2 n} \mathrm{d}\xi = (x = \xi\sqrt{n})$$

$$= 2 \int\limits_{|x|>\frac{\sqrt{n}}{2}} e^{-\pi x^2} \mathrm{d}x = O\left(\frac{1}{\sqrt{n}} e^{-\pi n/4}\right)$$

where we have used the trivial estimation

$$\alpha \int\limits_{x \geq \alpha} e^{-\pi x^2} \mathrm{d}x \leq \int\limits_{x \geq \alpha} x e^{-\pi x^2} \mathrm{d}x = O(e^{-\pi\alpha^2})$$

Thus, writing $a_n = \sum_k f^{1/n}(k) sinc(t-k)$, $b_n = f^{1/n}(t)$, we have proved

$$|b_n - a_n| \leq C \frac{1}{\sqrt{n}} e^{-\frac{\pi}{4}n}$$

Now, using the Middle Value Theorem is

$$\left| f(t) - a_n^n \right| = \left| b_n^n - a_n^n \right| \leq |b_n - a_n| \, n(1 + |b_n - a_n|)^{n-1}$$

Clearly

$$(1 + |b_n - a_n|)^{n-1} \leq \left(1 + \frac{C}{\sqrt{n}} e^{-\frac{\pi}{4}n}\right)^{n-1} = O(1)$$

since

$$(n-1) \log\left(1 + \frac{C}{\sqrt{n}} e^{-\frac{\pi}{4}n}\right) \sim n \frac{C}{\sqrt{n}} e^{-\frac{\pi}{4}n} \to 0$$

when n goes to infinity.

Therefore

$$\left| f(t) - a_n^n \right| \leq Kn \, |b_n - a_n| \leq Kn \frac{1}{\sqrt{n}} e^{-\frac{\pi}{4}n} \to 0$$

when $n \to \infty$ proving the result and moreover the uniform convergence under all compact set of $\mathbb{R}$. □

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